

A NOTE ON HOMOLOGICAL PROPERTIES OF NAKAYAMA ALGEBRAS

DAWEI SHEN

ABSTRACT. Using the resolution quiver for a connected Nakayama algebra, a fast algorithm is given to decide whether its global dimension is finite or not and whether it is Gorenstein or not. The latter strengthens a result of Ringel.

1. INTRODUCTION

Let A be a connected Nakayama algebra. Following [6, 4, 7], its *resolution quiver* is defined as follows: the vertex set is the set of non-isomorphic simple A -modules; there is a unique arrow from each simple A -module S to $\gamma(S) = \tilde{\tau} \operatorname{soc} P(S)$. Here, $P(S)$ is the projective cover of S and ‘soc’ is the socle of a module. If A has a simple projective module, denote by S_{inj} the unique simple injective A -module up to isomorphism. Then

$$\tilde{\tau}(S) = \begin{cases} \tau(S), & \text{if } S \text{ is not projective} \\ S_{\text{inj}}, & \text{otherwise} \end{cases}$$

for each simple A -module S , where τ is the Auslander-Reiten translation [1].

Let A be a connected Nakayama algebra. Denote by $R(A)$ its resolution quiver. It is known that each connected component of $R(A)$ has a unique cycle. For a cycle C in $R(A)$ with vertices S_1, S_2, \dots, S_m , the *weight* $w(C)$ of C is $\sum_{k=1}^m \frac{c_k}{n}$. Here, n is the number of non-isomorphic simple A -modules and c_k is the composition length of $P(S_k)$. It turns out that $w(C)$ is an integer and all cycles in $R(A)$ have the same weight [7]. The weight $w(C)$ is called the weight of the algebra A .

The resolution quiver is a very efficient tool for investigating the homological properties of Nakayama algebras. The Gorenstein projective modules for Nakayama algebras are described by resolution quivers [6]. Resolution quivers are also used to study the singularity categories of Nakayama algebras [8].

The resolution quiver of a connected Nakayama algebra gives a fast algorithm to decide whether it is Gorenstein or not and whether it is CM-free or not [6]. In this paper, we show that the resolution quiver of a connected Nakayama algebra also gives a fast algorithm to decide whether its global dimension is finite or not.

More precisely, we have the following.

Proposition 1.1. *Let A be a connected Nakayama algebra. Then A has finite global dimension if and only if its resolution quiver is connected and its weight is 1.*

As a consequence of Proposition 1.1, the resolution quiver is connected for a connected Nakayama algebra with finite global dimension.

For a connected Nakayama algebra, recall from [6] that a cycle in its resolution quiver is called *black* provided that the projective dimension of each simple module on this cycle is not equal to 1.

The following result strengthens [6, Proposition 5(a)].

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Proposition 1.2. *Let A be a connected Nakayama algebra with infinite global dimension. Then A is a Gorenstein algebra if and only if all cycles in its resolution quiver are black.*

Let A be a connected Nakayama algebra. Take a complete set $\{S_1, S_2, \dots, S_n\}$ of pairwise non-isomorphic simple A -modules. The *Cartan matrix* \mathbf{C}_A of A is an $n \times n$ matrix (c_{ij}) , where c_{ij} is the number of copies of S_i appearing in a composition series for the projective cover of S_j . Denote by \mathbf{C}_A^T the transpose of \mathbf{C}_A .

Denote by c the number of cycles and by b the number of black cycles in the resolution quiver of A .

The following result gives a connection between Cartan matrices and resolution quivers.

Proposition 1.3. *Let A be a connected Nakayama algebra.*

- (1) *The rank of \mathbf{C}_A is $n + 1 - c$.*
- (2) *If b is nonzero, then the rank of $(\mathbf{C}_A, \mathbf{C}_A^T)$ is $n + 1 - b$.*

The paper is organised as follows. The proofs of Proposition 1.1 and Proposition 1.2 are given in Section 2 and Section 3, respectively. In Section 4, we study the connection between Cartan matrices and resolution quivers for Nakayama algebras and prove Proposition 1.3.

2. RETRACTIONS AND RESOLUTION QUIVERS

Let A be a connected Nakayama algebra. Denote by $n = n(A)$ the number of non-isomorphic simple A -modules. Take a sequence (S_1, S_2, \dots, S_n) of pairwise non-isomorphic simple A -modules such that the radical of P_i is a factor module of P_{i+1} for $1 \leq i \leq n-1$ and the radical of P_n is a factor module of P_1 . Here, P_i is the projective cover of S_i . Denote by c_i the composition length of P_i . The *admissible sequence* for A is given by $\mathbf{c}(A) = (c_1, c_2, \dots, c_n)$; see [1, Chapter IV. 2].

Following [4], there exists a map $f_A: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ such that n divides $f_A(i) - (c_i + i)$. For $1 \leq i \leq n$, we have $\gamma(S_i) = S_{f_A(i)}$. Then for $1 \leq i, j \leq n$, there is an arrow $S_i \rightarrow S_j$ in the resolution quiver of A if and only if $f_A(i) = j$.

Suppose now that A is not selfinjective. If A has no simple projective modules, after possible cyclic permutations, we may assume that its admissible sequence is *normalized* [3], that is, $p(A) = c_1 = c_n - 1$. Here, $p(A)$ is the minimal integer among c_i . For convenience, if A has a simple projective module, its admissible sequence is always normalized.

Following [3], there exists a *left retraction* $\eta: A \rightarrow L(A)$, where $L(A)$ is a connected Nakayama algebra with admissible sequence $\mathbf{c}(L(A)) = (c'_1, c'_2, \dots, c'_{n-1})$ such that $c'_i = c_i - \left\lfloor \frac{c_i + i - 1}{n} \right\rfloor$ for $1 \leq i \leq n-1$. In particular, $n(L(A)) = n(A) - 1$. Here, $\lfloor x \rfloor$ is the largest integer not greater than a real number x . The corresponding sequence of simple $L(A)$ -modules is denoted by $(S'_1, S'_2, \dots, S'_{n-1})$.

We need the map $\pi: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n-1\}$ such that $\pi(i) = i$ for $1 \leq i \leq n-1$ and $\pi(n) = 1$.

Recall from [7, Lemma 2.1] the following result. For the convenience of the reader, we give a proof here.

Lemma 2.1. *Let A be a connected Nakayama algebra which is not selfinjective. Then $\pi f_A(i) = f_{L(A)}(i)$ for $1 \leq i \leq n-1$.*

Proof. Let $f_A(i) = j$ and $c_i + i = kn + j$ with $k \in \mathbb{N}$. For $1 \leq i \leq n-1$, we have

$$c'_i + i = c_i + i - \left\lfloor \frac{c_i + i - 1}{n} \right\rfloor = kn + j - \left\lfloor \frac{kn + j - 1}{n} \right\rfloor = k(n-1) + j. \quad (2.1)$$

It follows that $\pi f_A(i) = \pi(j) = f_{L(A)}(i)$. \square

Denote by γ' the map $\bar{\tau} \operatorname{soc} P(-)$ for $L(A)$. It follows from Lemma 2.1 that $\gamma'(S'_i) = S'_{\pi_{f_A}(i)}$ for $1 \leq i \leq n-1$. Hence the resolution quiver of $L(A)$ can be obtained from the resolution quiver of A just by “merging” the vertices S_1 and S_n .

Observe that $\gamma(S_n) = \gamma(S_1)$ if A has no simple projective modules and $\gamma(S_n) = S_1$ if A has a simple projective module. In particular, the vertices S_1 and S_n lie on the same connected component in the resolution quiver $R(A)$ of A . Then $R(A)$ and $R(L(A))$ have the same number of connected components. It follows that they have the same number of cycles since each connected component of a resolution quiver has a unique cycle.

Let C be a cycle with vertices $S_{x_1}, S_{x_2}, \dots, S_{x_s}$ in $R(A)$. The weight $w(C)$ of C is $\sum_{k=1}^s \frac{c_{x_k}}{n(A)}$ and the size of C is the number of vertices on C .

The following result strengthens [7, Lemma 2.2], where the connected Nakayama algebra is required to have no simple projective modules.

Lemma 2.2. *Let A be a connected Nakayama algebra which is not selfinjective. Then there exists a weight preserving bijection between the set of cycles in $R(A)$ and the set of cycles in $R(L(A))$. Moreover, if A has no simple projective modules or the simple projective A -module does not lie on a cycle in $R(A)$, then the bijection also preserves the size.*

Proof. If A has no simple projective modules, then the bijection follows from [7, Lemma 2.2]. We may assume that A has a simple projective module S_n .

Let C be a cycle with vertices $S_{x_1}, S_{x_2}, \dots, S_{x_s}$ in $R(A)$. Assume $x_{i+1} = f_A(x_i)$. Here, we identify x_{s+1} with x_1 . Let $c_{x_i} + x_i = k_i n + x_{i+1}$ with $k_i \in \mathbb{N}$. Then

$$w(C) = \frac{\sum_{i=1}^s c_{x_i}}{n} = \sum_{i=1}^s k_i.$$

It follows from (2.1) that for $x_i < n$, we have $c'_{x_i} + x_i = k_i(n-1) + x_{i+1}$.

There exist two cases:

Case 1: S_n does not lie on C . Then $S'_{x_1}, S'_{x_2}, \dots, S'_{x_s}$ form a cycle C' in $R(L(A))$.

Observe that $c'_{x_i} + x_i = k_i(n-1) + x_{i+1}$ for $1 \leq i \leq s$. Then

$$\sum_{i=1}^s c'_{x_i} = (n-1) \sum_{i=1}^s k_i,$$

Therefore, $w(C) = w(C')$.

Case 2: S_n lies on C . Since the admissible sequence of A is normalized, we have $x_s = n$ and $x_1 = 1$. It follows that $S'_{x_1}, S'_{x_2}, \dots, S'_{x_{s-1}}$ form a cycle C'' in $R(L(A))$.

Observe that $k_s = 1$ and $c'_{x_i} + x_i = k_i(n-1) + x_{i+1}$ for $1 \leq i \leq s-1$. Then

$$\sum_{i=1}^{s-1} c'_{x_i} = (n-1) \sum_{i=1}^{s-1} k_i + n-1 = (n-1) \sum_{i=1}^s k_i.$$

Therefore, $w(C) = w(C'')$.

We have shown that there exists an injective weight preserving map from the set of cycles in $R(A)$ to the set of cycles in $R(L(A))$. Since $R(L(A))$ and $R(A)$ have the same number of cycles, this map is also surjective. This finishes our proof. \square

Recall from [3, Theorem 3.8] that there exists a sequence of left retractions

$$A = A_0 \xrightarrow{\eta_0} A_1 \xrightarrow{\eta_1} A_2 \longrightarrow \dots \longrightarrow A_{r-1} \xrightarrow{\eta_{r-1}} A_r \quad (2.2)$$

such that each A_i is a connected Nakayama algebra, each $\eta_i: A_i \rightarrow A_{i+1}$ is a left retraction and A_r is selfinjective; the global dimension of A is finite if and only if A_r is simple.

For a connected Nakayama A , denote by $c(A)$ the number of cycles and by $w(A)$ the weight of a cycle in $R(A)$. We mention that $w(A)$ is an integer and all cycles in $R(A)$ have the same weight; see [7].

We now prove Proposition 1.1.

Proof of Proposition 1.1. Applying Lemma 2.2 to (2.2) repeatedly, we obtain $c(A) = c(A_r)$ and $w(A) = w(A_r)$. Observe that A_r is simple if and only if $c(A_r) = 1$ and $w(A_r) = 1$. Then the global dimension of A is finite if and only if $c(A) = 1$ and $w(A) = 1$. Each connected component of a resolution quiver has a unique cycle. Then $c(A) = 1$ if and only if the resolution quiver of A is connected. \square

3. TWO MAPS ON SIMPLE MODULES

Let A be a connected Nakayama algebra. For an A -module M , denote by $\text{pd } M$ its projective dimension and by $\text{id } M$ its injective dimension. Recall that the *syzygy* $\Omega(M)$ of M is the kernel of its projective cover $p: P(M) \rightarrow M$. Dually, the *cosyzygy* $\Omega^{-1}(M)$ of M is the cokernel of its injective envelope $i: M \rightarrow I(M)$.

It is known that A has a simple projective module if and only if it has a simple injective module. In this case, denote by S_{inj} the unique simple injective A -module up to isomorphism. Then there exists map $\tilde{\tau}$ defined by

$$\tilde{\tau}(S) = \begin{cases} \tau(S), & \text{if } S \text{ is not projective} \\ S_{\text{inj}}, & \text{otherwise} \end{cases}$$

for each simple A -module S , where τ the Auslander-Reiten translation [1].

Take a complete set \mathcal{S} of pairwise non-isomorphic simple A -modules. Recall from [4, 6] that there exist two maps $\gamma, \psi: \mathcal{S} \rightarrow \mathcal{S}$ given by

$$\gamma(S) = \tilde{\tau} \text{soc } P(S) \text{ and } \psi(S) = \tilde{\tau}^{-1} \text{top } I(S).$$

for each S in \mathcal{S} . Here, ‘soc’ is the socle and ‘top’ is the top of a module.

The map γ determines the resolution quiver for A . Denote by A^{op} the opposite algebra of A and by γ^{op} the map $\tilde{\tau} \text{soc } P(-)$ for A^{op} . Then $D\psi(S) = \gamma^{\text{op}}(DS)$ for each S in \mathcal{S} , where D is the usual dual for finitely generated A -modules. Hence the resolution quiver of A^{op} is isomorphic to the quiver determined by the map ψ .

The following terminology is taken from [6]. A simple A -module S is called γ -black provided that $\text{pd } S$ is not equal to 1; it is called γ -cyclic provided that $\gamma^m(S) = S$ for some integer $m > 0$. Dually, one can define ψ -black and ψ -cyclic simple A -modules.

We need the following lemma.

Lemma 3.1. *Let S and T be simple A -modules.*

- (1) *If S is not projective, then $\psi(T) = S$ if and only if T is a composition factor in $\Omega^2(S)$.*
- (2) *If $\text{pd } \psi(S)$ is odd, then $\text{pd } S$ is odd and $\text{pd } S \leq \text{pd } \psi(S) - 2$.*
- (3) *S is ψ -cyclic if and only if $\text{pd } S$ is not odd.*
- (4) *A has infinite global dimension if and only if the set of ψ -cyclic simple A -modules is exactly the set of simple A -modules of infinite projective dimension.*

Proof. If A has no simple projective modules, then the arguments follow from [5, Section 3]. We mention that they are valid for any connected Nakayama algebra. \square

We need the following.

Lemma 3.2. *Let M be an indecomposable A -module. Then either $\text{id } M \leq 1$ or $\text{soc } \Omega^{-2}(M) = \psi(\text{soc } M)$.*

Proof. This is dual to [6, Lemma 2]; see also [4]. \square

The following lemma provides a connection between the maps γ and ψ .

Lemma 3.3. *A simple A -module S is γ -black if and only if $\psi\gamma(S) = S$.*

Proof. Suppose that S is a γ -black simple A -module. If S is projective, then by definition $\psi\gamma(S) = S$. If $\text{pd } S \geq 2$, then by Lemma 3.2 we have $\gamma(S) = \text{soc } \Omega^2(S)$. It follows from Lemma 3.1(1) that $\psi\gamma(S) = S$.

Suppose $\psi\gamma(S) = S$. If S is projective, then S is γ -black. If S is not projective, then it follows from Lemma 3.1(1) that $\gamma(S)$ is a composition factor in $\Omega^2(S)$. In particular, $\Omega^2(S)$ is nonzero and thus $\text{pd } S \geq 2$. \square

Recall that a cycle in the resolution quiver of A is called black provided that each vertex on this cycle is γ -black.

We have the following observation.

Proposition 3.4. *Let C be a cycle in the resolution quiver of A . Then the following statements are equivalent.*

- (1) *The vertices of C form a ψ -cycle.*
- (2) *Each vertex on C is ψ -cyclic.*
- (3) *C is a black cycle.*

Proof. (1) \implies (2) This is obvious.

(2) \implies (3) By Lemma 3.1(3) each ψ -cyclic simple A -module is γ -black.

(3) \implies (1) Assume that the vertices of C are S_1, S_2, \dots, S_m with $S_{i+1} = \gamma(S_i)$ for $i \geq 1$. Here, we identify S_{m+1} with S_1 .

Since C is a black cycle, each S_i is γ -black and each $\text{id } S_i$ is not odd for $i \geq 1$. It follows from Lemma 3.3 that $\psi\gamma(S_i) = \psi(S_{i+1}) = S_i$.

We claim that each $\text{pd } S_i$ is not odd and thus each S_i is ψ -cyclic by Lemma 3.1(3). Since $\psi(S_{i+1}) = S_i$ for $i \geq 1$, it follows that S_m, S_{m-1}, \dots, S_1 form a ψ -cycle.

For the claim, suppose to the contrary that the projective dimension of some S_i is odd. Then by Lemma 3.1(2) S_{i+1} is odd and $\text{pd } S_{i+1} \leq \text{pd } S_i - 2$. By induction, we infer that $\text{pd } S_{i+k} \leq \text{pd } S_i - 2k$ for $k \geq 1$. This is a contradiction. \square

Suppose that global dimension of A is infinite. In particular, A has no simple projective modules. The following lemma describes projective A -modules of finite injective dimension and injective A -modules of finite projective dimension.

Lemma 3.5. *Let A be a connected Nakayama algebra with infinite global dimension, and let P be an indecomposable projective A -module and I be an indecomposable injective A -module.*

- (1) *The injective dimension of P is infinite if and only if P is a nontrivial submodule of $P(S)$ with S a γ -cyclic simple A -module.*
- (2) *The projective dimension of I is infinite if and only if I is a nontrivial factor module of $I(S)$ with S a ψ -cyclic simple A -module.*

Proof. (1) “ \Leftarrow ” If the injective dimension of P is infinite, then the injective dimension of its cosyzygy $\Omega^{-1}(P)$ is also finite. It follows that $\Omega^{-1}(P)$ contains at least one composition factor with infinite injective dimension.

We claim that $\Omega^{-1}(P)$ contains at most one composition factor which is γ -cyclic. It follows from Lemma 3.1(4) that $\Omega^{-1}(P)$ contains precisely one composition factor which is γ -cyclic. Since P is a nontrivial submodule of $P(T)$ for each composition factor T in $\Omega^{-1}(P)$, we infer that P is a nontrivial submodule of $P(S)$ with S a γ -cyclic simple A -module.

For the claim, observe that $\text{soc } P(T) = \text{soc } P$ and $\gamma(T) = \gamma(\text{top } P)$ for each composition factor T in $\Omega^{-1}(P)$. Since the composition factors in $\Omega^{-1}(P)$ have the same image under the map γ , at most one of them is γ -cyclic.

“ \implies ” Suppose that P is a nontrivial submodule of $P(S)$ with S a γ -cyclic simple A -module. By the previous claim, one can show that there exists only one composition factor in $\Omega^{-1}(P)$ which is γ -cyclic, namely S . Since the injective dimension of S is infinite by Lemma 3.1(4), the injective dimension of $\Omega^{-1}(P)$ is infinite. Then the injective dimension of P is infinite.

(2) This is dual to (1). \square

Recall that an Artin algebra A is called a *Gorenstein algebra* if both $\text{id } A$ and $\text{pd } DA$ are finite. Here, D is the usual dual for finitely generated A -modules.

We are now ready to prove Proposition 1.2. Indeed, using the maps γ and ψ as above, several characterizations are given to decide whether a connected Nakayama algebra with infinite global dimension is Gorenstein or not. It strengthens [6, Proposition 5(a)].

Proposition 3.6. *Let A be a connected Nakayama algebra with infinite global dimension. Then the following statements are equivalent.*

- (1) *A is a Gorenstein algebra.*
- (2) *Each γ -cyclic simple A -module is γ -black.*
- (3) *Each ψ -cyclic simple A -module is ψ -black.*
- (4) *The set of γ -cyclic simple A -modules is exactly the set of ψ -cyclic simple A -modules.*

Proof. (1) \implies (2). Let S be a γ -cyclic simple A -module. By Lemma 3.5(1) the projective cover of S has no nontrivial projective submodules. Then the projective dimension of S greater than 1 and thus S is γ -black.

Similarly, we have (1) \implies (3).

(2) \implies (4). Following Proposition 3.4 the set of γ -cyclic simple A -modules is contained in the set of ψ -cyclic simple A -modules. By [7, Corollary 2.3] the two sets have the same finite number of modules. Then they must coincide.

(4) \implies (2). Since each ψ -cyclic simple A -module is γ -black, it follows that each γ -cyclic simple A -module is γ -black.

Similarly, one can prove that (3) and (4) are equivalent.

(2) + (3) \implies (1). By Lemma 3.5 all projective A -modules have finite injective dimension and all injective A -modules have finite projective dimension. Then A is a Gorenstein algebra. \square

We mention that the global dimension condition in Proposition 3.6 cannot be omitted; see [6, Example 2].

4. CARTAN MATRICES AND RESOLUTION QUIVERS

In this section, we study the connection between the Cartan matrix and the resolution quiver for a fixed connected Nakayama algebra A .

Denote by $n = n(A)$ the number of non-isomorphic simple A -modules. Take a complete set $\{S_1, S_2, \dots, S_n\}$ of pairwise non-isomorphic simple A -modules. The Cartan matrix $\mathbf{C}_A = (c_{ij})$ of A is an $n \times n$ matrix, where c_{ij} is the number of copies of S_i appearing in a composition series for the projective cover of S_j .

Recall that two $n \times n$ integer matrix \mathbf{X} and \mathbf{Y} are \mathbb{Z} -equivalent provided that there exist invertible integer matrices \mathbf{P} and \mathbf{Q} such that $\mathbf{P}\mathbf{X}\mathbf{Q} = \mathbf{Y}$. For an $n \times n$ integer matrix, its *Smith normal form* is the $n \times n$ diagonal integer matrix

$$\text{diag}(d_1, d_2, \dots, d_r, 0, \dots, 0)$$

where $d_1, d_2, \dots, d_r \in \mathbb{N}^+$ and d_i divides d_{i+1} for $1 \leq i \leq r-1$. The Smith normal form always exists and is unique; it is \mathbb{Z} -equivalent to the original matrix.

Denote by $c(A)$ the number of cycles and by $w(A)$ the weight of a cycle in the resolution quiver of A .

The following result provides a connection between the Cartan matrix and the resolution quiver for A .

Proposition 4.1. *Let A be a connected Nakayama algebra. Then the Smith normal form of its Cartan matrix \mathbf{C}_A is the diagonal matrix*

$$\text{diag}(1, \dots, 1, w(A), 0, \dots, 0)$$

with $c(A) - 1$ zeros on the diagonal. In particular, the rank of \mathbf{C}_A is $n(A) + 1 - c(A)$.

Proof. Recall (2.2) in Section 2. For $1 \leq i \leq r-1$, it is easy to show that the Cartan matrix \mathbf{C}_{A_i} of A_i and the block diagonal matrix $\text{diag}(1, \mathbf{C}_{A_{i+1}})$ are \mathbb{Z} -equivalent.

By Lemma 2.2 we have $c(A_i) = c(A_{i+1})$ and $w(A_i) = w(A_{i+1})$. Then it suffices to prove the assertion for the selfinjective algebra A_r .

Assume that A is a connected selfinjective Nakayama algebra. Denote by m its radical length. Then the admissible sequence of A is (m, m, \dots, m) . It is routine to show that the resolution quiver of A consists of $\gcd(m, n)$ cycles of the same weight $w = \frac{m}{\gcd(m, n)}$. Here, ‘gcd’ is the greatest common divisor.

Let $m = kn + r$ with $k \in \mathbb{N}$ and $1 \leq r \leq n$. After possible permutations on simple A -modules, the Cartan matrix \mathbf{C}_A is a circulant matrix given by

$$c_{ij} = \begin{cases} k + 1, & \text{if } 0 \leq i - j < r \text{ or } j - i > n - r \\ k, & \text{otherwise.} \end{cases}$$

It follows from [9] that the Smith normal form of the Cartan matrix \mathbf{C}_A is the diagonal matrix $\text{diag}(1, \dots, 1, w, 0, \dots, 0)$ with $\gcd(m, n) - 1$ zeros on the diagonal. Then the rank of \mathbf{C}_A is $n + 1 - \gcd(m, n)$. This finishes our proof. \square

Remark 4.2. Use the notation in the Proposition 4.1.

- (1) Following [2, Theorem 6], the global dimension of A is finite if and only if the determinant of the Cartan matrix \mathbf{C}_A is 1. In fact, by [3, Proposition 2.2(5)] left retractions also preserve the determinants of Cartan matrices. Then the determinant of the Cartan matrix \mathbf{C}_A is $w(A)$ if the resolution quiver of A is connected; see also [2, Lemma 2]. This provides another proof of Proposition 1.1.
- (2) Denote by A^{op} the opposite algebra of A . Then the Cartan matrix of A^{op} is the transpose of \mathbf{C}_A . Since \mathbf{C}_A and its transpose \mathbf{C}_A^T have the same Smith normal form, it follows that $c(A) = c(A^{\text{op}})$ and $w(A) = w(A^{\text{op}})$; see also [8, Proposition 5.2].

Let X be a subset of $\{S_1, S_2, \dots, S_n\}$. There exists a $n \times 1$ vector ξ_X associated with X , where the i -th entry of ξ_X is 1 if S_i is in X and the i -th entry of ξ_X is 0 if S_i is not in X for $1 \leq i \leq n$. Denote by $\mathbf{1}$ the $n \times 1$ vector $(1, \dots, 1)^T$.

We have the following observation.

Proposition 4.3. *Let A be a connected Nakayama algebra. Denote by Γ the set of cycles and by $B\Gamma$ the set of black cycles in the resolution quiver of A .*

- (1) *The vectors $\{\xi_C\}_{C \in \Gamma}$ are maximal linearly independent solutions to the linear system $\mathbf{C}_A \xi = w(A)\mathbf{1}$.*
- (2) *The vectors $\{\xi_C\}_{C \in \Gamma}$ are the entire nonnegative integer solutions to the linear system $\mathbf{C}_A \xi = w(A)\mathbf{1}$.*
- (3) *The vectors $\{\xi_E\}_{E \in B\Gamma}$ are maximal linearly independent solutions to the linear system $\mathbf{C}_A^T \xi = \mathbf{C}_A \xi = w(A)\mathbf{1}$.*

Proof. (1) Since C is a cycle in the resolution quiver, for $1 \leq i \leq n$ the simple A -module S_i appears exactly $w(A)$ times in the direct sum $\oplus_{S \in C} P(S)$. It follows that $\mathbf{C}_A \xi_C = w(A)\mathbf{1}$. Since the vectors $\{\xi_C\}_{C \in \Gamma}$ have disjoint support, they are linearly independent. Then we obtain $c(A)$ linearly independent solutions to the linear system $\mathbf{C}_A \xi = w(A)\mathbf{1}$. By Proposition 4.1 the number of these solutions is $n + 1 - \text{rank } \mathbf{C}_A$. Therefore, the solutions $\{\xi_C\}_{C \in \Gamma}$ are maximal.

(2) This follows from (1).

(3) Let E be a cycle in the resolution quiver of A . By Proposition 3.4 the cycle E is black if and only if the vertices of E form a ψ -cycle. By (2) the vertices of E form a ψ -cycle if and only if $\mathbf{C}_A^T \xi_E = w(A)\mathbf{1}$. Then the vectors $\{\xi_E\}_{E \in B\Gamma}$ are linearly independent solutions to the linear system $\mathbf{C}_A^T \xi = \mathbf{C}_A \xi = w(A)\mathbf{1}$.

Denote by Ψ is the set of ψ -cycles for A . We claim that the vectors $\{\xi_C\}_{C \in \Gamma \cup \Psi}$ are linearly independent. For a solution ξ to the desired linear system, by (1) we have $\xi = \sum_{C \in \Gamma} a_C \xi_C = \sum_{D \in \Psi} b_D \xi_D$. Observe that the intersection $\Gamma \cap \Psi$ is the set of black cycles. Since the vectors $\{\xi_C\}_{C \in \Gamma \cup \Psi}$ are linearly independent, we have $\xi = \sum_{C \in B\Gamma} a_C \xi_C$. This proves (3).

For the claim, let $\sum_{C \in \Gamma \cup \Psi} a_C \xi_C = 0$. For a non-black cycle $C \in \Gamma$, it follows from Proposition 3.4 that there exists some S_i on C which is not ψ -cyclic. Observe that the i -th entry of $\sum_{C \in \Gamma \cup \Psi} a_C \xi_D$ is a_C . It follows that $a_C = 0$ for each non-black cycle $C \in \Gamma$. Since the vectors $\{\xi_C\}_{C \in \Psi}$ have disjoint support, $a_C = 0$ for each ψ -cycle C . Therefore, the vectors $\{\xi_C\}_{C \in \Gamma \cup \Psi}$ are linearly independent. \square

Denote by $b(A)$ be the number of black cycles in the resolution quiver of A .

We have the following.

Corollary 4.4. *Let A be a connected Nakayama algebra.*

- (1) $b(A)$ is nonzero if and only if $\text{rank} \begin{pmatrix} \mathbf{C}_A^T \\ \mathbf{C}_A \end{pmatrix} = \text{rank} \begin{pmatrix} \mathbf{C}_A^T & \mathbf{1} \\ \mathbf{C}_A & \mathbf{1} \end{pmatrix}$.
- (2) If $b(A)$ is nonzero, then $b(A) = n(A) + 1 - \text{rank}(\mathbf{C}_A, \mathbf{C}_A^T)$.

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DEPARTMENT OF MATHEMATICS, SHANGHAI KEY LABORATORY OF PMMP, EAST CHINA NORMAL UNIVERSITY, SHANGHAI 200241, P. R. CHINA
E-mail address: dwshen@math.ecnu.edu.cn